Emil Artin
Intertwining Representation Theory and Cohomology

Eric M. Friedlander

Support varieties for algebraic groups

Michael Artin
Representation theory

$G$ a group and $V$ a vector space

$G \times V \rightarrow V, \quad (g, v) \mapsto g \circ v$

Conditions:

$(g_1 \cdot g_2) \circ v = g_1 \circ (g_2 \circ v)$,

$g \circ (a \cdot v + b \cdot w) = a \cdot (g \circ v) + b \cdot (g \circ w)$.

What sort of groups?
What sorts of vector spaces?
Lie theory

Sophus Lie
1842 - 1899

Perspective of geometry and differential equations

“continuous transformation groups”

acting continuously, e.g. on a real or complex vector space $V$

Lie theory: Understand these continuous representations of $G$ in terms of representations of $\mathfrak{g} = \text{Lie}(G)$. 
Algebraic groups

Fact: Each simple complex Lie group can be viewed as zero locus of further polynomial equations inside some $GL_N(\mathbb{C})$.

Claude Chevalley

Algebraic groups over a field $k$

$GL_n$ is zero locus inside $\mathbb{A}^{n^2+1}$ of $det(x_{i,j}) \cdot z = 1$
What is an algebraic representation of an algebraic group?

If $V$ is finite dimensional, $V = k^\oplus N$, then $G \times V \to V$ is algebraic (a.k.a. “rational representation”) if each matrix coefficient as a function of $G$ is algebraic (i.e., in $k[G]$).

**Example:** Let $G = GL_2$ act on polynomials of degree $n$ in 2 variables $k[x, y]_n$. Explicitly, we can write this as

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \circ x^i y^{n-i} = (ax + by)^i(cx + dy)^{n-i}
$$

Equivalently: comodule structure $\Delta : V \to V \otimes k[G]$, so that $\Delta(v) = \sum v_i \otimes f_i$ with $g \circ v = \sum_i f_i(g)v_i$. 
Example: $\mathbb{F}_p$ – field of $p$-elements. (Every non-zero element has an inverse; add 1 to itself $p$ times and the answer is 0.)

Example: $q = p^d$, $d > 0$. There is a unique finite field $\mathbb{F}_q$ of order $q$.

Example: If $F$ is a field of characteristic $p > 0$, then so is $F(x)$.

If $X \subset \mathbb{A}^N$ is the zero locus of polynomial equations with coefficients in $\mathbb{F}_q$, then sending $(x_1, \ldots, x_N) \in \mathbb{A}^N$ to $(x_1^q, \ldots, x_N^q) \in \mathbb{A}^N$ sends points of $X$ to points of $X$.

Key point: $(a + b)^p = a^p + b^p$ in characteristic $p$.

Frobenius map $F^q : X \rightarrow X$. 
Wildness for finite groups, char $p > 0$

$k$-linear actions of $\mathbb{Z}/p \times \mathbb{Z}/p$ on $k$-vector space $V$ correspond to actions of

$$k[g, h]/(g^p = 1 = h^p) \cong k[x, y]/(x^p, y^p), \quad g = x+1, \ h = y+1$$

**Example:** Indecomposable, not irreducible

$\mathbb{Z}/p \times \mathbb{Z}/p$ has wild representation type (for $p > 2$).
Intertwining Representation Theory and Cohomology

Eric M. Friedlander

Support varieties for algebraic groups

Solomon Lefschetz

1884 - 1972

Applications of algebraic topology to algebraic geometry

(classical) algebraic geometry

Characteristic \( p \) STINKS!
Cohomology

**Definition**

If $G \times M \to M$ is a $G$-action on the $k$-vector space $M$, then

$$H^0(G, M) = M^G = \{ m \in M; g \circ m = m, \forall g \in G \}.$$

$$H^i(G, M) = (R^i(H^0(G, -))(M).$$

If every indecomposable $G$-module is irreducible, then $H^i(G, M) = 0$, $i > 0$.

$H^1(G, M)$ equals the group of equivalence classes of short exact sequences $0 \to M \to E \to k \to 0$ of $G$-modules (i.e., extensions of $k$ by $M$).
Cohomology and Geometry

Daniel Quillen

1940 – 2011

Spectrum of cohomology of a finite group $G$

$\text{Spec}(H^*(G, k))$, affine algebraic variety

Extension of Quillen by J. Carlson et al: use spirit of Quillen to study representations of a finite group $G$. 
Failure of Lie theory over fields of characteristic $p > 0$

**EXAMPLE** $SL_2$ action on the homogeneous polynomials of degree precisely $p$ in two variables: $V = k[x, y]^p$. Inside $V$, there is a 2-dimensional subrepresentation $W \subset V$ consisting of polynomials linear in $x^p, y^p$. There is no splitting of $W \subset V$ as representations of $SL_2$.

$V$ is *indecomposable*, but not *irreducible*.

Much WORSE news:
If action of $SL_2$ on vector space $V$ factors through $F : SL_2 \to SL_2$, then Lie algebra action is *trivial* (because differential $d(F) = 0 : \mathfrak{sl}_2 \to \mathfrak{sl}_2$).
Functors and group schemes

Alexander Grothendieck
1928 - 2014

Functorial point of view

A group scheme over $k$ is a \textit{functor}

$$(\text{comm. } k\text{-alg}) \to \text{(groups)}.$$  

Replace the Lie algebra of $G$ by "\textit{infinitesimal neighborhoods of the identity}”, so called “Frobenius kernels” $G_{(r)}$. 

Frobenius kernels

One can view $G_r \subset G$ as a representable subfunctor of $G$
(comm. $k$–alg) $\to$ (groups), \( R \mapsto \ker\{ F^r : G(R) \to G(R) \} \).

Example

$GL_N(r)$ has coordinate algebra $k[X_{i,j}]/(X_{i,j}^{p^r} - \delta_{i,j})$, a finite dimensional, commutative, local $k$-algebra; multiplication of dual $kGL_N(r)$ is given by the multiplication of $GL_N$.

$kG_r$ is always a f. dim, co-commutative Hopf algebra.

Given a representation $G \times V \to V$, this structure is faithfully reflected by the collection of structures $\{ G_r \times V \to V \}$.
The additive group $\mathbb{G}_a$

**Definition**

$\mathbb{G}_a$: (comm. $k$-alg) $\rightarrow$ (abelian groups); $\mathbb{G}_a(R) = R^+$. $k[\mathbb{G}_a] = k[T]$; group structure determined by comultiplication

$$\Delta : k[T] \rightarrow k[T] \otimes k[T], \quad T \mapsto (T \otimes 1) + (1 \otimes T).$$

**Lemma**

A $\mathbb{G}_a$-action on a $k$-vector space $V$ is naturally equivalent to the following data:

An infinite sequence of $p$-nilpotent, pairwise commuting: operators $u_0, u_1, u_2, u_3 \ldots : V \rightarrow V$ such that for any $v \in V$ all but finitely many $u_i(v)$ are 0.
Varieties for $\mathbb{G}_a$-modules

**Definition**

The cohomological variety $V^{coh}(\mathbb{G}_a) \equiv \text{Spec}_{cont} H^*(\mathbb{G}_a, k)$.

The 1-parameter subgroup variety $V(\mathbb{G}_a) \equiv \{ \psi : \mathbb{G}_a \rightarrow \mathbb{G}_a \}$.

**Proposition**

$V^{coh}(\mathbb{G}_a) \simeq \mathbb{A}^\infty \simeq V(\mathbb{G}_a)$.

**Definition**

$V^{coh}(\mathbb{G}_a)_M \equiv \{ p \subset H^*(\mathbb{G}_a, k) : p \supset \text{ann}(H^*(\mathbb{G}_a, M)) \}$.

$V(\mathbb{G}_a)_M \equiv \{ \psi : \mathbb{G}_a \rightarrow \mathbb{G}_a : \text{such that NOT all blocks of size p for action at } \psi \}$.
Support varieties for $\mathbb{G}_a$

For $M$ finite dimensional, $V^{coh}(\mathbb{G}_a)_M = V(\mathbb{G}_a)_M \subset \mathbb{A}^\infty$.

- Many “standard” properties including $V(\mathbb{G}_a)_M = \{0\}$ if $M$ is injective, $V(\mathbb{G}_a)_M = \mathbb{A}^\infty$ if $M = k$.

- “Mock injective” modules: there exist (necessarily infinite dimensional) $\mathbb{G}_a$-modules $M$ which are not injective, but $V(\mathbb{G}_a)_M = \{0\}$.

- Know exactly which subvarieties $X \subset \mathbb{A}^\infty$ are of form $X = V(\mathbb{G}_a)_M$ for some finite dimensional $\mathbb{G}_a$-module $M$.

- Lots of interesting questions about which $X \subset \mathbb{A}^\infty$ are of form $X = V(\mathbb{G}_a)_M$ for an arbitrary $\mathbb{G}_a$-module $M$. 
Other algebraic groups $G$

Cohomology NOT very useful in general. For example, $H^*(G, k)$ is trivial for $G$ a simple algebraic group.

Will describe a theory using 1-parameter subgroups $\psi : \mathbb{G}_a \rightarrow G$ which has many useful properties.

For $G$ unipotent (e.g., $U_N \subset GL_N$), then study of 1-parameter subgroups leads to cohomological calculations.
1-parameter subgroups for infinitesimal kernels $G(r)$

Andrei Suslin

Joint work: Quillen’s geometry extends to Frobenius kernels.

computations for $H^*(G(r), k)$ in terms of the variety of $V(G(r))$ of infinitesimal 1-parameter subgroups $G_{a(r)} \to G(r)$.

Theorem [Suslin-F-Bendel] $V^{coh}(G(r))_M$ can be identified with the variety $V(G(r))_M$. 

Action of $G$ on $M$ at a 1-parameter subgroup

**Theorem**

Assume that $G$ is a linear algebra group of exponential type. The ind-variety $V(G)$ of 1-parameter subgroups of $G$ is $\simeq$ variety $\mathcal{C}_\infty(\mathcal{N}_p(\text{Lie}(G)))$ consisting of finite sequences of $p$-nilpotent, pair-wise commuting elements of $\text{Lie}(G)$:

$$\{B \in \mathcal{C}_\infty(\mathcal{N}_p(\text{Lie}(G)))\} \sim V(G), \quad B \mapsto \mathcal{E}_B.$$

**Definition**

The action on a rational $G$-module $M$ at the 1-parameter subgroup $\mathcal{E}_B : \mathbb{G}_a \to G$ is the action of the $p$-nilpotent operator

$$\sum_{s \geq 0} (\mathcal{E}_{B_s})^*(u_s) \in kG.$$
Formulation of support variety $V(G)_M$

### Definition

The support variety $V(G)_M \subset V(G)$ of $M$ is the subset of those $B \in C_\infty(\mathcal{N}_p(Lie(G)))$ such that $\psi_B$ has some block of size $< p$.

- For $M$ finite dimensional, $V(G)_M$ carries the same information as the earlier considered $V(G_r)_M$ for $r >> 0$.
- For $G = \mathbb{G}_a$, $V(\mathbb{G}_a)_M \simeq V^{coh}(\mathbb{G}_a)_M$.
- For $G = U_N$, $V^{coh}(U_N)_M$ is much less informative than $V(U_N)_M$.
- Leads to interesting classes of mock injective and mock trivial $G$-modules.
- Can compute some examples of the form $V(G)_{G/H}$.
“Classical properties” of \( M \mapsto V(G)_M \)

<table>
<thead>
<tr>
<th>Theorem</th>
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<tbody>
<tr>
<td>1. <strong>Tensor product</strong>: ( V(G)_{M \otimes N} = V(G)_M \cap V(G)_N ).</td>
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<td>2. <strong>Two out of three</strong>: If ( 0 \to M_1 \to M_2 \to M_3 \to 0 ), then the support variety ( V(G)_{M_i} ) of one of the ( M_i ) is contained in the union of the support varieties of the other two.</td>
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| 3. **For the Frobenius twist** \( M^{(1)} \) of \( M \), \[
V(G)_{M^{(1)}} = \{ \mathcal{E}_{(B_0, B_1, B_2 \ldots)} \in V(G) : \mathcal{E}_{(B_1^{(1)}, B_2^{(1)}, \ldots)} \in V(G)_M \}. \]
| 4. For any \( r > 0 \), the restriction of \( M \) to \( kG(r) \) is injective (equivalently, projective) if and only if the intersection of \( V(G)_M \) with the subset \( \{ \psi_B : B_s = 0, s > r \} \subset V(G) \) equals \( \{ \mathcal{E}_0 \} \). |
Strategy: For computations of $H^*(U_N(r), k)$, $H^*(U_N, k)$:

- (F-Suslin) give a means of construction of cohomology classes.
- (Suslin-F-Bendel) give detection of cohomology modulo nilpotents.
- Use the descending central series

$$U_N = \Gamma_1 \supset \Gamma_2 \supset \cdots \subset \Gamma_N = \{e\}$$

with each subquotient a product of $\mathbb{G}_a$’s.

- Key tool is the $T_N$-equivariant Lyndon-Hochschild-Serre spectral sequence along with the action of the Steenrod algebra.
LHS Spectral sequence

**Key technique** for computation is the $T_N$-equivariant Hochschild-Serre spectral sequence

$$E_2^{*,*} = H^*(U_N/\Gamma_{v-1}, k) \otimes H^*(\Gamma_{v-1}/\Gamma_v, k) \Rightarrow H^*(U_N/\Gamma_v, k)$$

for terms of the descending central series for $U_N$.

**Compute differentials** using the Steenrod algebra: for example, $d^{0,2p^j}_{2p^j+1}((x^{(i)}_{s,t})^{p^j})$ equals

$$\sum_{t=1}^{v-1}(x^{(i)}_{s,s+t})^{p^j} \otimes y^{(i+1+j)}_{s+t,s+v} - (x^{(i)}_{s+t,s+v})^{p^j} \otimes y^{(i+1+j)}_{s,s+t}).$$

We conclude the relation

$$(x^{(i)}_{s,s+1})^{p^j+1} \cdot (x^{(i)}_{s_1,s+2})^{i+1+j} - (x^{(i)}_{s+1,s+2})^{p^j+1} \cdot (x^{(i)}_{s,s+1})^{i+1+j} \quad 0 \leq j.$$
Calculations of cohomology

**Theorem**

For $p \geq N - 1$, $H^*((U_N)_r, k)$ modulo nilpotents is given by *explicit construction* augmenting $k[V_r(U_N)]$.

*Similar statement of terms $U_N/\Gamma_v$ of lower central series.*

**Remark**

This improves [Suslin-F-Bendel] in that we can compare for increasing $r$, take the *limit as $r$ goes to $\infty$.*

**Theorem**

$\text{Spec}_{\text{cont}} H^\bullet(U_3, k)$ is determined by the image of $H^\bullet(U_3/\Gamma, k) \to H^\bullet(U_3, k)$. 
Continuous prime ideal spectrum

**Definition**

\[ V^{coh}(G) \equiv \lim_{\rightarrow r} \text{im}\{\text{Spec } H^\bullet(G(r), k) \to \text{Spec } H^\bullet(G, k)\}. \]

**Example**

\[ H^*(\mathbb{G}_a, k) = S^*(x^{(i)}, i \geq 1) \otimes \Lambda^*(y^{(i)}, i \geq 0), \] so that

\[ V^{coh}(\mathbb{G}_a) \cong \mathbb{A}^\infty. \]

**Proposition**

For \( p \geq 3 \), \( H^\bullet(U_N, k) \) embeds in \( \lim_{\leftarrow r} H^\bullet(U_N(r), k) \).

**Proposition**

There exists a *natural, surjective map*

\[ \text{Proj } V(G)_M \to \text{Proj } V^{coh}(G)_M. \]